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A Random Choice Method for Two-Dimensional Steady Supersonic Shock Wave Diffraction Problems*†

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A random choice method for the numerical solution of steady, supersonic, two-dimensional plane and three-dimensional axisymmetric gas flows is presented. The random choice method uses exact solutions of Riemann problems and sampling techniques. It is applicable to steady, supersonic flows in more than one dimension because these flows are described by a hyperbolic system of conservation laws in two independent variables. The method is applied to a variety of supersonic shock wave diffraction problems and compared to solutions obtained with the method of characteristics. The main advantages of the method presented are its general applicability and its sharp resolution of discontinuities in the flow. © 1984 Academic Press, Inc.

1. INTRODUCTION

The study of shock wave diffraction patterns, i.e., the deflection of a shock wave through interaction with an obstacle or another wave, is of considerable importance in fluid dynamic research. The basic types of shock wave diffraction: attached shocks, regular reflection, single Mach reflection, complex Mach reflection, and double Mach reflection, have been extensively studied through physical experiments in shock tubes and wind tunnels (see, e.g., the work of Ben-Dor and Glass [1]). They constitute the basic components of more complicated gas flows over aerodynamic structures.

Numerical models have proven to be powerful tools of analysis in the study and simulation of complicated fluid dynamic flows. If properly validated, these models can provide an accurate description of the flow field. In addition, they provide a way to validate new physical and mathematical hypotheses, thereby helping to understand complex wave interactions.

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Two-dimensional plane and three-dimensional axisymmetric gas flows can be described, under suitable assumptions, by a hyperbolic system of conservation laws of the form

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_r + \mathbf{g}(\mathbf{w})_z = \boldsymbol{h}(\mathbf{w}, r) \tag{1.1}$$

with appropriate initial and boundary conditions. In this paper we are interested in steady (time invariant) flows. These flows are governed by the reduced form of system (1.1) obtained by assuming that $\mathbf{w}_t = 0$,

$$\mathbf{f}(\mathbf{w})_r + \mathbf{g}(\mathbf{w})_z = \mathbf{h}(\mathbf{w}, r). \tag{1.2}$$

In certain physical situations, numerical solutions of these equations can be obtained as the large time limit of system (1.1), but here we will solve system (1.2) directly. Depending on the physics of the flow considered, system (1.2) can be globally hyperbolic (supersonic flow), mixed hyperbolic and elliptic (transonic flow), or globally elliptic (subsonic flow). This in turn determines the methods of solution that are applicable. Many interesting steady state shock wave diffraction problems are supersonic, in which case the system (1.2) is hyperbolic. For example, the deflection of a supersonic stream by an abrupt or gradual ramp can be globally supersonic; this type of flow involves the interaction of a shock with a Mach wave.

There is a well-known mathematical analogy between two-dimensional supersonic steady flows governed by system (1.2) and one-dimensional time-dependent flows. These latter flows are governed by the reduced form of system (1.1) obtained by assuming that $\mathbf{w}_z = 0$,

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_r = \mathbf{h}(\mathbf{w}, r). \tag{1.3}$$

The essence of the mathematical analogy is that both problems are described by hyperbolic systems of conservation laws in two independent variables. There is also a qualitatively physical analogy which we will now describe by use of a simple example (see Courant and Friedrichs [4]). Figure 1.1 shows the one-dimensional time-dependent flow produced by the abrupt starting and stopping of a piston, while

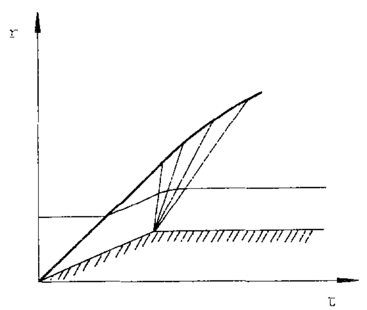

FIG. 1.1. One-dimensional time-dependent flow produced by the abrupt starting and stopping of a piston.

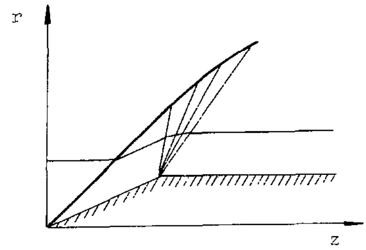

FIG. 1.2. Two-dimensional steady supersonic flow around a wedge.

Fig. 1.2 shows two-dimensional steady supersonic flow around a wedge. In the analogy between these flows, the particle paths, the shock path, and the characteristic fan (sound waves) of the time-dependent one-dimensional flow correspond, respectively, to the streamlines, the oblique shock, and the Prandtl–Meyer fan (Mach waves) of the two-dimensional steady supersonic flow. Similarly, two-dimensional, axisymmetric, time-dependent flows and three-dimensional, axisymmetric, supersonic steady flows are analogous.

(This analogy extends to higher dimensional flow patterns. In this paper, however, we make the assumption of axisymmetry because the numerical methods to be described are applicable only to partial differential equations with two independent variables. For example, there are difficulties with extending the random choice method to time-dependent two-dimensional flows and hence to truly three-dimensional supersonic steady flow. But by extension of the analogy the solution of the steady supersonic Riemann problems can be used, e.g., in a two-dimensional shock tracking scheme to solve fully three-dimensional steady supersonic flow problems.)

Because systems (1.2) and (1.3) are mathematically analogous they can be solved, in principle, using the same numerical procedures. In particular, if the z-coordinate in system (1.2) may be considered as the time-like coordinate, the system (1.2) can be solved using a time-like marching procedure similar to those used for the solution of time-dependent flows. Many researchers in the field have used marching techniques for solving supersonic steady flows (e.g., Moretti *et al.* [10] and Wardlaw *et al.* [13], among others). In this work we develop a random choice method to be used as the marching procedure.

Numerical solutions of hyperbolic system of conservation laws must satisfy the following criteria: (a) the computed solution must be sufficiently accurate in the smooth part of the flow; (b) discontinuities such as contact lines and shock should remain sharp and should be transported at the correct speed; and (c) strong discontinuities should be computed stably. The methods that have been most widely used for finding numerical solutions of discontinuous hyperbolic problems can be subdivided in three main categories: shock capturing, shock tracking (shock fitting), and adaptive grid methods. Shock capturing techniques are fixed grid methods in which the interaction of waves can be handled automatically. In contrast with these techniques, shock tracking methods follow the discontinuities explicitly. At the discontinuity the integral form of the system of conservation laws is solved to propagate the discontinuity; the discontinuities also form the boundaries of regions of smooth flow, in which the differential form of the conservation laws is solved. Adaptive grid methods automatically select moving grids to follow propagating waves, or use local mesh refinement on stationary grids. Also included in this category is the method of characteristics, wich is the adaptive method *par excellence*.

Many computational methods for solving gas flow problems are based on approximating the problem with a number of more elementary flow problems, called Riemann problems. The solution of these Riemann problems are important because they provide an explicit and elementary class of solutions which contain extensive information about wave interaction. They are the basic constructive step in the random choice method and they provide the key input for shock tracking methods.

The random choice method is a shock capturing technique for computing solutions of hyperbolic systems of conservation laws. It consists of approximating the solution at each time step by a piecewise constant state and advancing to the next time step by solving the local Riemann problems formed by the constant states on adjacent spatial mesh intervals. The value of the approximate solution over each mesh interval of the new time step is taken to be the exact solution evaluated at a randomly chosen point. The main advantages of the method are its general applicability and its ability to resolve discontinuities and sharp interfaces without incurring over- and undershooting phenomena. The random choice method was introduced by Glimm [7] for homogeneous system of conservation laws; it was developed into a numerical method by Chorin [2], who made extensive use of it for computations of combustion problems. Extensions of this method that apply to inhomogeneous hyperbolic systems of conservation laws have been developed by Sod [12], Fok [5], Glimm *et al.* [8], and Glaz and Liu [6]. Chorin [3] has extended the random choice method to apply to situations where the boundaries of the computational domain change in time; Goodman [9] has analyzed this method as applied to initial-boundary value problems for one-dimensional conservation laws.

The purpose of this paper is to present a random choice method for supersonic steady flow based on the solution of the Riemann problem described by Plohr [11]. We apply it to a number of flow problems and compare the results to those obtained by the method of characteristics. The plan of the paper is as follows. In Section 2 we discuss the equations describing steady gas flows. In Section 3 the method of characteristics and the random choice method are briefly reviewed. In Section 4 we describe the solution of the Riemann problem used in the random choice method. In Section 5 we present numerical results. Finally, in Section 6 some conclusions are drawn.

2. Two-Dimensional Gas Flow

The equations describing the axisymmetric flow of an inviscid, compressible gas may be written in the form

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_r + \mathbf{g}(\mathbf{w})_z = \mathbf{h}(\mathbf{w}, r), \tag{2.1}$$

where

$$\mathbf{w} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \qquad \mathbf{h}(\mathbf{w}, r) = -\frac{d-2}{r} \begin{bmatrix} m \\ \dfrac{m^2}{\rho} \\ \dfrac{mn}{\rho} \\ \dfrac{m}{\rho}(E+p) \end{bmatrix},$$

$$\mathbf{f}(\mathbf{w}) = \begin{bmatrix} m \\ \dfrac{m^2}{\rho} + p \\ \dfrac{mn}{\rho} \\ \dfrac{m}{\rho}(E+p) \end{bmatrix}, \qquad \text{and} \qquad \mathbf{g}(\mathbf{w}) = \begin{bmatrix} n \\ \dfrac{mn}{\rho} \\ \dfrac{n^2}{\rho} + p \\ \dfrac{n}{\rho}(E+p) \end{bmatrix},$$

Here ρ is the mass density of the fluid, m and n are the r- and z-components of momentum density ($m = \rho u$ and $n = \rho v$, where u and v are the r- and z-components of fluid velocity), E is the total energy density, p is the thermodynamic pressure, and d is the dimension of space. The four component equations of the system (2.1) express, respectively, the conservation of mass, Newton's law in the r and z directions, and the conservation of energy. These equations are supplemented with the thermodynamic equation of state

$$p = p(\rho, E - (m^2 + n^2)/2\rho).$$

We will assume the gas to be polytropic, so that $p(\rho, e) = (\gamma - 1)e$ for some constant $\gamma > 1$. System (2.1) is subject to appropriate initial and boundary conditions.

We are interested in steady state and one-dimensional time-dependent solutions of the system (2.1). For a steady state solution $\mathbf{w}$ is independent of time, so that $\mathbf{w}$ satisfies the equations

$$\mathbf{f}(\mathbf{w})_r + \mathbf{g}(\mathbf{w})_z = \mathbf{h}(\mathbf{w}, r). \tag{2.2}$$

For a one-dimensional time-dependent solution $\mathbf{w}$ is independent of the z coordinate, so that $\mathbf{w}$ satisfies the equations

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_r = \mathbf{h}(\mathbf{w}, r). \tag{2.3}$$

(In this latter case we may drop the z component of Newton's equation, so that $\mathbf{w}$, $\mathbf{f}(\mathbf{w})$, and $\mathbf{h}(\mathbf{w}, r)$ are vectors with only three components; also, $d - 2$ in the definition of $\mathbf{h}$ should be replaced with $d-1$ to maintain the interpretation of d as the spatial dimension.)

If system (2.2) is hyperbolic and z is a time-like direction it is possible to solve the initial-boundary value problem (2.2) using a marching procedure along the z coordinate, in a fashion similar to the methods used to solve system (2.3). For example, and explicit finite difference analogue of system (2.2) may be written

$$\mathbf{g}(\mathbf{w}_i^{n+1}) = \mathbf{g}(\mathbf{w}_i^n) + \Delta z[\mathbf{h}(\mathbf{w}_i^n, r) - (\partial\mathbf{f}/\partial r)_i^n] = \mathbf{G}_i^n, \tag{2.4}$$

where $\mathbf{w}_i^n$ is the numerical solution at $z = n\Delta z$ and $r = i\Delta r$, Δz and Δr being the time and space increments, and where $(\partial\mathbf{f}/\partial r)_i^n$ is the finite difference approximation to $\mathbf{f}(\mathbf{w})_r$, which depends on the particular differencing utilized. For simplicity let us dispense with the space and time indices. Writing system (2.4) in terms of its components we then obtain

$$n = \mathbf{G}^{(1)}, \qquad \frac{mn}{\rho} = \mathbf{G}^{(2)},$$

$$\frac{n^2}{\rho} + p = \mathbf{G}^{(3)}, \qquad \text{and} \qquad \frac{n}{\rho}\left(\frac{m^2+n^2}{2\rho} + \frac{\gamma}{\gamma-1}p\right) = \mathbf{G}^{(4)}, \tag{2.5}$$

where $\mathbf{G}^{(k)}$ denotes the kth component of the vector $\mathbf{G}$. This system of equations can be reduced to solving a quadratic equation for the density, viz.

$$\left[\frac{1}{2}\left(\frac{\mathbf{G}^{(2)}}{\mathbf{G}^{(1)}}\right)^2 - \frac{\mathbf{G}^{(4)}}{\mathbf{G}^{(1)}}\right]\rho^2 + \frac{\gamma}{\gamma-1}\mathbf{G}^{(3)}\rho - \frac{1}{2}\frac{\gamma+1}{\gamma-1}\mathbf{G}^{(1)2} = 0. \tag{2.6}$$

Thus an explicit finite difference marching calculation for the solution of system (2.2) consists of calculating certain functions of the primitive variables at successive steps in the z direction and then solving a quadratic equation to obtain the primitive variables.

3. The Method of Characteristics and the Random Choice Method

In this section we briefly describe the method of characteristics and the random choice method for the steady flow of an inviscid, compressible, polytropic gas. The latter method is based on the solution of the Riemann problem to be described in Section 4.

The method of characteristics [4] applies to smooth gas flows, so that it is permissible to rewrite system (2.2) in its characteristic form

$$A(\mathbf{U})\,\mathbf{U}_r + B(\mathbf{U})\,\mathbf{U}_z = C(\mathbf{U}, r), \tag{3.1}$$

where

$$U = \begin{pmatrix} \rho \\ u \\ v \\ p \end{pmatrix}, \qquad C(\mathbf{U}, r) = -\frac{d-2}{r} \begin{pmatrix} \rho u \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A(\mathbf{U}) = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & \rho u & 0 & 1 \\ 0 & 0 & \rho u & 0 \\ -c^2 u & 0 & 0 & u \end{pmatrix}, \quad \text{and} \quad B(\mathbf{U}) = \begin{pmatrix} v & 0 & \rho & 0 \\ 0 & \rho v & 0 & 0 \\ 0 & 0 & \rho v & 1 \\ -c^2 v & 0 & 0 & v \end{pmatrix}.$$

The characteristic equation for system (3.1) is

$$(u - \lambda v)^2 [(u - \lambda v)^2 - c^2(1 + \lambda^2)] = 0, \tag{3.2}$$

which has roots

$$\lambda_0 = u/v \qquad \text{and} \qquad \lambda_\pm = \frac{uv \pm c^2 (M^2 - 1)^{1/2}}{v^2 - c^2}. \tag{3.3}$$

If $\xi = \tan^{-1}(v/u)$ and $\beta = \sin^{-1} M^{-1}$ denote the flow and Mach angles, the characteristic slopes can be written as

$$\lambda_0 = \cot \xi \qquad \text{and} \qquad \lambda_\pm = \cot(\xi \mp \beta). \tag{3.4}$$

A characteristic whose slope is λ_0 is a streamline, while one whose slope is $\lambda_\pm$ is a Mach line. Along characteristics the fluid quantities are constrained by ordinary differential equations: along streamlines

$$\rho u \, du + \rho v \, dv + dp = 0 \qquad \text{and} \qquad dp - c^2 \, d\rho = 0, \tag{3.5}$$

while along Mach lines

$$\mp d\xi + \frac{(M^2 - 1)^{1/2}}{M^2} \frac{dp}{\rho c^2} + \frac{d-2}{r} \frac{\cos \xi}{M \sin(\xi \mp \beta)} dz = 0. \tag{3.6}$$

A finite difference analogue of Eqs. (3.4) to (3.6) on a characteristic mesh can be used to obtain numerical solutions of system (3.1). We note, however, that since the equations apply only to smooth flows, any use of the method of characteristics for problems involving shocks requires special adaptive techniques at the shock. Thus to apply the method of characteristics to flow over a wedge, where an attached shock develops, it is necessary to track the shock trajectory and apply the equations for an oblique shock' explicitly. Furthermore, the method of characteristics calculates the fluid quantities only at the nodes of the characteristic net, which is not rectangular (see Fig. 5.3), so that an interpolation procedure is necessary to obtain values on a rectangular grid (e.g., for the purpose of plotting isopycnics). To obtain our results below we modified an implementation of the method of characteristics adapted for flow over a wedge by Hoffman [14], and used a routine from the NCAR library to perform the interpolation.

The random choice method is a numerical scheme for solving hyperbolic systems of conservation laws which is based on a constructive proof due to Glimm [7]. Consider the hyperbolic system

$$\mathbf{w}_t + \mathbf{f}(\mathbf{w})_x = 0 \tag{3.7}$$

subject to the initial conditions

$$\mathbf{w}(x, t=0) = \mathbf{w}_0(x) \tag{3.8}$$

for all x. (Here, e.g., $\mathbf{w}$ and $\mathbf{f}(\mathbf{w})$ could describe one-dimensional gas flow.) We introduce a space–time grid defined by mesh lengths Δx and Δt. The solution is to be calculated at points of the form ($x = i\Delta x$, $t = n\,\Delta t$), where i and n are integers. Let us denote $\mathbf{w}(i\,\Delta x, n\,\Delta t)$ by $\mathbf{w}_i^n$. Because we impose the initial conditions (3.3) we know the values of $\mathbf{w}_i^n$ for all i; thus to specify the scheme it suffices to describe how $\mathbf{w}_i^{n+1}$ is calculated once $\mathbf{w}_{i-1}^n$, $\mathbf{w}_i^n$, and $\mathbf{w}_{i+1}^n$ are known. Consider the following initial value problem, known as Riemann problem,

$$\bar{\mathbf{w}}_t + \mathbf{f}(\bar{\mathbf{w}})_x = 0 \tag{3.9}$$

subject to the initial conditions

$$\begin{aligned} \bar{\mathbf{w}}(x, t = n\,\Delta t) &= \mathbf{w}_i^n \qquad \text{for} \quad x < (i + \tfrac{1}{2})\,\Delta x, \\ &= \mathbf{w}_{i+1}^n \qquad \text{for} \quad x \geqslant (i + \tfrac{1}{2})\,\Delta x. \end{aligned} \tag{3.10}$$

Assume that we can obtain the solution $\bar{\mathbf{w}}$ of this problem. Also assume that we have been given an equidistributed sequence η^n of real numbers in the interval $[-\frac{1}{2}, \frac{1}{2}]$. (We have used the Van der Corput equidistributed sequence.) Then if $\eta^{n+1} \geqslant 0$ we define

$$\mathbf{w}_1^{n+1} = \bar{\mathbf{w}}((i + \eta^{n+1})\,\Delta x, (n+1)\,\Delta t), \tag{3.11}$$

while if $\eta^{n+1} < 0$ we define $\mathbf{w}_i^{n+1}$ in an analogous way in terms of the solution of the Riemann problem formed using $\mathbf{w}_{i-1}^n$ and $\mathbf{w}_i^n$ at $x = (i - \frac{1}{2})\,\Delta x$. By continuing this process the approximate solution is defined. Glimm proved that under certain assumptions the family of approximate solutions obtained by successively refining the grid will converge to a weak solution of Eq. (3.1). (The most important assumption, one that we will impose in our numerical implementation, is that the Courant–Friedrichs–Lewy (CFL) condition be satisfied, viz., that $s_{\max}\,\Delta t \leqslant \Delta x/2$, where $s_{\max}$ is the maximum wave speed of the solution.) This procedure may be also used to obtain numerical solutions of Eq. (3.7) if it is numerically feasible to solve the corresponding Riemann problems. Of course, the solution of a Riemann problem is often much simpler to obtain than the solution of a general initial value problem. In the next section we review the solution of the Riemann problem for steady supersonic flow.

To apply the random choice method to system (2.2), we first note that there is no difficulty in replacing (3.7) with the more general equation

$$\mathbf{f}(\mathbf{w})_r + \mathbf{g}(\mathbf{w})_z = 0, \tag{3.12}$$

so long as this equation is hyperbolic and the Riemann problems may be solved. The addition of a source term, as in (2.2), requires only a minor modification of the method and can be effected in a number of ways [12, 5, 8, 6]. For the results in this paper we have used the splitting method of Sod [12], in which two steps are used to advance the solution in "time" by Δz: first the Riemann problem for (3.12), i.e., (2.2) with source terms removed, is solved and sampled, as in the random choice method; then the ordinary differential equation

$$\mathbf{g}(\mathbf{w})_z = \mathbf{h}(\mathbf{w}, r), \tag{3.13}$$

i.e., (2.2) with convection terms removed and r fixed, is solved, using the result of the first step as the initial condition.

For the problems we solve below it is also necessary to handle "moving" boundaries: just as in the case of one-dimensional flow influenced by the movement of a piston, steady supersonic flow is influenced by the change in the radial position of a wedge surface as a function of the z coordinate. To this end we have implemented the extension of the random choice scheme to general initial–boundary value problems due to Chorin [3] (see also Goodman [9]). In this method the boundary position is of the form $r^n_{\text{bdry}} = (i + \frac{1}{2})\Delta r$ at each step $z = n\,\Delta z$. From the known state (say) $\mathbf{w}^n_{i+1}$ in the interior of the flow a virtual state $\mathbf{w}^n_i$ inside the boundary is constructed, and the corresponding Riemann problem is solved and sampled. (For steady supersonic gas flow with Neumann boundaries, the virtual state has the same density and energy density as the interior state, and the virtual momentum density is obtained by reflecting the interior momentum density with respect to the sloping boundary. Thus the solution of the Riemann problem will have a middle state with momentum density parallel to the boundary.) The position r^{n+1}_{bdry} of the boundary for $z = (n+1)\Delta z$ is obtained as follows. The slope dr_{bdry}/dz is prescribed for each z, so we may define

$$\bar{r} = r^n_{\text{bdry}} + \Delta z \left(\frac{dr_{\text{bdry}}}{dz}\right)_{z = n\Delta z}. \tag{3.14}$$

Then if $\bar{r} < (i + \eta^{n+1})\Delta r$, $r^{n+1}_{\text{bdry}} = r^n_{\text{bdry}} - \Delta r$; while if $\bar{r} > (i + 1 + \eta^{n+1})\Delta r$, $r^{n+1}_{\text{bdry}} = r^n_{\text{bdry}} + \Delta r$; and otherwise $r^{n+1}_{\text{bdry}} = r^n_{\text{bdry}}$. Thus this method passively tracks the position of the boundary in accordance with the sampling procedure, so that even the boundary exhibits the random fluctuations inherent in the method. The same passive tracking is used to follow any shocks present in the initial data; this allows the use of a contour plotter that does not interpolate across the shock (cf. Figs. 5.1 and 5.6).

4. The Riemann Problem for Steady Supersonic Flow

A Riemann problem for a hyperbolic system of conservation laws

$$\mathbf{f}(\mathbf{w})_r + \mathbf{g}(\mathbf{w})_z = 0 \tag{4.1}$$

is an initial value problem for which the initial data at $z = 0$ consists of two constant states $\mathbf{w}_{\text{left}}$ and $\mathbf{w}_{\text{right}}$ separated by a jump discontinuity at $r = r_{\text{jump}}$. Because the equation and the initial data are invariant under the scaling transformation $(r, z) \to (r_{\text{jump}} + \alpha(r - r_{\text{jump}}), \alpha z)$, the solution depends only on the polar angle θ, where $\cot\theta = (r - r_{\text{jump}})/z$. Consequently a Riemann problem can be reduced to solving a system of ordinary differential equations for the smooth parts of the solution and a system of nonlinear equations for the discontinuous parts of the solution. For steady supersonic flow of a polytropic gas the differential equation can be integrated explicitly.

Here we summarize the result of the analysis of the Riemann problem for steady supersonic flow of polytropic gases (for details see Plohr [11]). The solution of a Riemann problem is constructed from three elementary waves, a backward wave, a slip line, and a forward wave. Backward and forward waves are of two types, rarefactions and shocks. The following describes the properties of these waves. For convenience we take $r_{\text{jump}} = 0$, and we specify the state of the gas with the density ρ, pressure p, Mach number M, and flow angle $\xi = \tan^{-1}(v/u)$.

(1) *Rarefaction waves* are smooth solutions. They are two families, forward and backward, corresponding to the + and − cases below. In the r–z plane a rarefaction is a wedge separating two constant states $\mathbf{w}_{\text{ahead}}$ and $\mathbf{w}_{\text{behind}}$. Inside the rarefaction the state of the gas is constant along radial lines; at the angle

$$\theta = \xi \mp \sin^{-1} M^{-1} \tag{4.2}$$

the following equations are satisfied:

$$\begin{gathered} \pm\varepsilon = \pm\xi_{\text{ahead}} + \nu_\gamma(M) - \nu_\gamma(M_{\text{ahead}}), \\ M = \left(\frac{2}{\gamma - 1}\right)^{1/2} \left[\frac{\rho}{\rho_{\text{ahead}}} \frac{p_{\text{ahead}}}{p} \left(1 + \frac{\gamma - 1}{2} M_{\text{ahead}}^2\right) - 1\right]^{1/2}, \end{gathered} \tag{4.3}$$

and

$$\rho = \rho_{\text{ahead}} \left(\frac{p}{p_{\text{ahead}}}\right)^{1/\gamma}.$$

Here ν_γ is the Prandtl–Meyer function

$$\nu_\gamma(M) = \int_1^M \frac{(M^2 - 1)^{1/2}}{1 + ((\gamma - 1)/2) M^2} \frac{dM}{M} = \lambda_\gamma^{-1} \tan^{-1}(\lambda_\gamma (M^2 - 1)^{1/2}) - \tan^{-1}(M^2 - 1)^{1/2} \tag{4.4}$$

with $\lambda_\gamma = [(\gamma - 1)/(\gamma + 1)]^{1/2}$. For rarefactions, $p_{\text{behind}} \leqslant p_{\text{ahead}}$.

(2) *Shock waves* are discontinuous solutions, also of two families, forward and backward. A shock with discontinuity along the ray at angle θ separates two constant states $\mathbf{w}_{\text{ahead}}$ and $\mathbf{w}_{\text{behind}}$ that are related by the following equations:

$$\pm\xi_{\text{behind}} = \pm\xi_{\text{ahead}} + \sin^{-1}\left[M_{\text{behind}}^{-1}\left(1+\frac{\gamma+1}{2\gamma}\left(\frac{p_{\text{ahead}}}{p_{\text{behind}}}-1\right)\right)^{1/2}\right]$$

$$-\sin^{-1}\left[M_{\text{ahead}}^{-1}\left(1+\frac{\gamma+1}{2\gamma}\left(\frac{p_{\text{behind}}}{p_{\text{ahead}}}-1\right)^{1/2}\right],$$

$$M_{\text{behind}} = \left(\frac{2}{\gamma-1}\right)^{1/2}\left[\frac{\rho_{\text{behind}}}{\rho_{\text{ahead}}}\frac{p_{\text{ahead}}}{p_{\text{behind}}}\left(1+\frac{\gamma-1}{2}M_{\text{ahead}}^2\right)-1\right)\right]^{1/2}, \quad (4.5)$$

and

$$\rho_{\text{behind}} = \rho_{\text{ahead}}\frac{1+((\gamma+1)/2\gamma)\left(\dfrac{p_{\text{behind}}}{p_{\text{ahead}}}-1\right)}{1+((\gamma-1)/2\gamma)\left(\dfrac{p_{\text{behind}}}{p_{\text{ahead}}}-1\right)}.$$

For shocks, $p_{\text{behind}} > p_{\text{ahead}}$.

(3) *Slip lines* are discontinuous solutions of only one family. A slip line with discontinuity along the ray at an angle θ separates constant states $\mathbf{w}_{\text{left}}$ and $\mathbf{w}_{\text{right}}$, that have the same flow angle $\xi_{\text{left}} = \theta = \xi_{\text{right}}$ and the same pressure $p_{\text{left}} = p_{\text{right}}$ but arbitrary Mach number and densities.

The solution of a Riemann problem consists of a backward shock or rarefaction on the left, a forward shock or rarefaction on the right, and a slip line in between. Let $\mathbf{w}_{\text{left}}$ and $\mathbf{w}_{\text{right}}$ denote the initial states of the gas on the two sides of the jump, and let $\xi*$ and $p*$ denote the common values of the flow angle and the pressure on either side of the slip line. It is convenient to introduce the following functions:

$$\kappa(\eta) = \eta^{1/\gamma} \qquad \text{if} \quad \eta \leqslant 1,$$

$$= \frac{1+((\gamma+1)/2\gamma)(\eta-1)}{1+((\gamma-1)/2\gamma)(\eta-1)} \qquad \text{if} \quad \eta > 1,$$

$$\mu(\eta; M_0) = \left(\frac{2}{\gamma-1}\right)^{1/2}\left[\eta^{-1}\kappa(\eta)\left(1+\frac{\gamma-1}{2}M_0^2\right)-1\right]^{1/2} \quad (4.6)$$

and

$$\chi(\eta; M_0) = \nu_\gamma(\mu(\eta; M_0)) - \nu_\gamma(M_0) \qquad \text{if } \eta \leqslant 1,$$

$$= \sin^{-1}\left[\mu(\eta; M_0)^{-1}\left(1+\frac{\gamma+1}{2\gamma}(\eta^{-1}-1)\right)^{1/2}\right]$$

$$-\sin^{-1}\left[M_0^{-1}\left(1+\frac{\gamma+1}{2\gamma}(\eta-1)\right)^{1/2}\right] \qquad \text{if } \eta > 1.$$

From the properties of shock and rarefaction waves above we see that

$$\xi * = \xi_{\text{right}} + \chi(p*/p_{\text{right}}; M_{\text{right}}) \qquad \text{and} \qquad -\xi * = -\xi_{\text{left}} + \chi(p*/p_{\text{left}}; M_{\text{left}}). \qquad (4.7)$$

Adding these equations yields the implicit equation

$$\xi_{\text{left}} - \xi_{\text{right}} = \chi(p*/p_{\text{left}}; M_{\text{left}}) + \chi(p*/p_{\text{right}}; M_{\text{right}}) \qquad (4.8)$$

for $p*$.

Once this equation has been solved for $p*$ we may calculate $\xi*$ using either of the two previous equations, and we may obtain the Mach numbers and densities on either side of the slip line using the functions μ and κ. From this information the complete structure of the forward and backward waves may be determined by using the equations defining shock and rarefaction solutions.

5. Numerical Results

We present the results of numerical tests of our method applied to the flow of an inviscid, compressible, polytropic gas over wedges, cones, and compression ramps. The results of the first test are compared to numerical results obtained with the method of characteristics. The numerical solutions are described by means of contour plots of the density (isopycnics) in the r–z plane.

In the first test we studied planar flow over a wedge. The initial conditions along the r axis consist of uniform flow parallel to the z axis with $M = 3$. Neumann

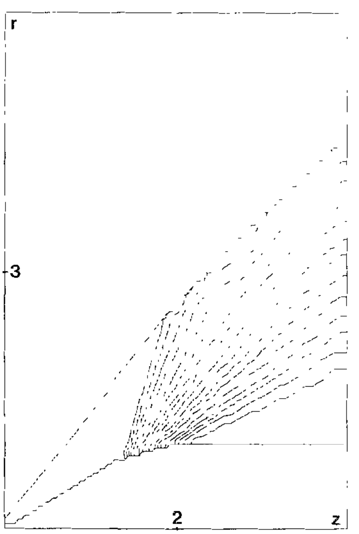

FIG. 5.1. Isopycnics for plane ($d = 2$) shock wave diffraction as obtained with the random choice method. The incident Mach number is $M = 3$ and the wedge angle is 30°.

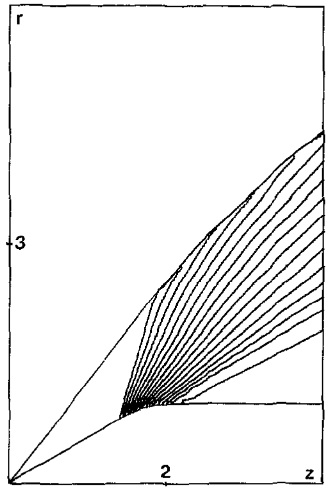

FIG. 5.2. Isopycnics for plane ($d = 2$) shock wave diffraction as obtained with the method of characteristics. The incident Mach numbers is $M = 3$ and the wedge angle is 30°.

(reflecting) boundary conditions are imposed on the boundaries above and below. The bottom boundary "moves" as a function of the "time" z, following a trajectory that forms a wedge with an angle of 30°. The second corner of the wedge is not sharp, but rather is smoothed into a circular arc. Because the incoming flow is supersonic, a shock forms at the tip of the wedge; this shock is diffracted by the Prandtl–Meyer fan formed at the second corner of the wedge. Figure 5.1 shows the results obtained with the random choice method with 120 spatial grid points. For comparison purposes, the results obtained using the method of characteristics are

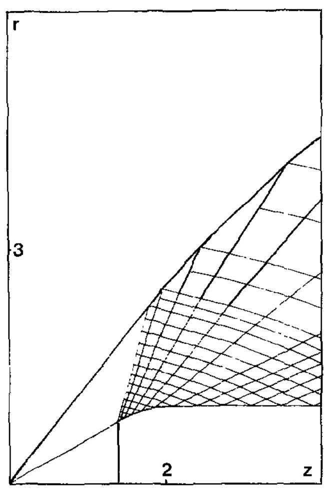

FIG. 5.3. Characteristic net constructed by the method of characteristics for the plane shock wave diffraction in Fig. 5.2.

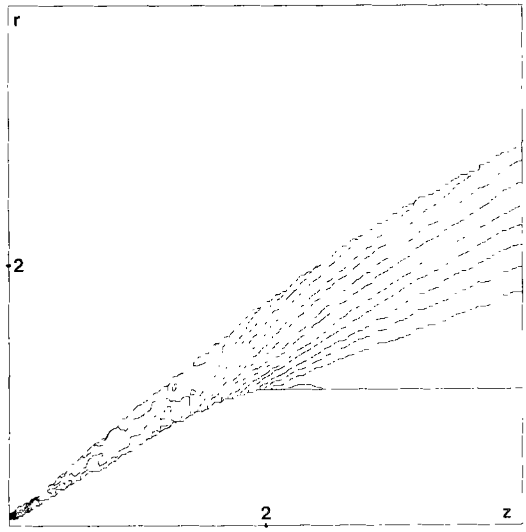

FIG. 5.4. Isopycnics for conic flow ($d = 3$) as obtained with the random choice method. The incident Mach number is $M = 3$ and the cone angle is 30°.

shown in Figs. 5.2 and 5.3. Figure 5.2 shows the isopycnics, while Fig. 5.3 shows the characteristic net constructed by this method. We remark that the results obtained with the method of characteristics can be considered to be very close to the exact solution. We can observe that the results of the random choice method, on the average, are very close to those of the method of characteristics.

In the second numerical experiment we studied axisymmetric flow over a conical wedge. This test is similar to the first, except that the dimension d is taken to be 3

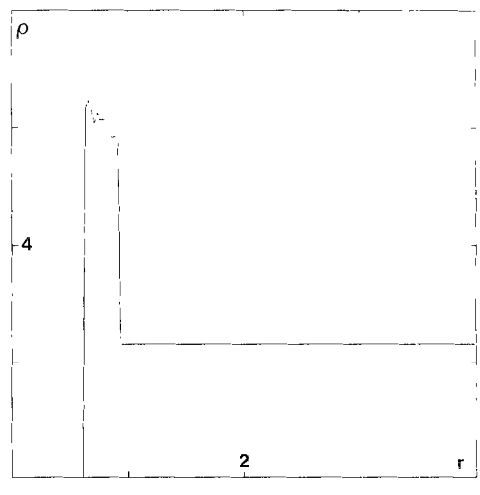

FIG. 5.5. Density versus radius at $z = 1$ for the flow over the cone shown in Fig. 5.4.

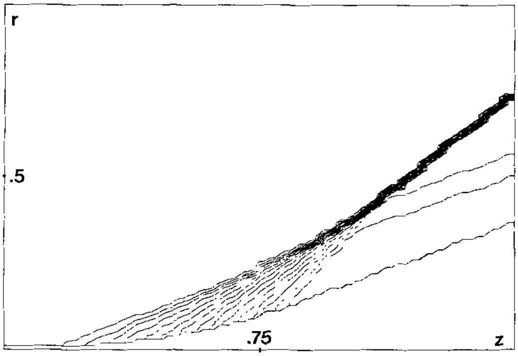

FIG. 5.6. Isopycnics for a compression corner as obtained with the random choice method. The incident Mach number is $M = 3$ and the turning angle is 20°.

instead of 2. Since (2.2) now has a nonzero inhomogeneous term, the random choice method must be modified; here we use the modification due to Sod [12] with 160 spatial grid points. Figure 5.4 shows the results obtained with this method. The results reflect the fact that the conical shock is swept back more than the corresponding plane shock. (Note that the vertical scale in Fig. 5.4 differs from that in Fig. 5.1.) Between the shock and the tip of the cone the density should increase slightly because of the source term. This feature can only barely be seen in the contour plot, being obscured by random fluctuations, but the density cross section at $z = 1$ shown in Fig. 5.5 reflects the correct behavior.

In the third test we studied planar flow over a compression ramp. The initial conditions along the r axis again consist of uniform flow parallel to the z axis with $M = 3$, and Neumann (reflecting) boundary conditions are imposed on the boundaries

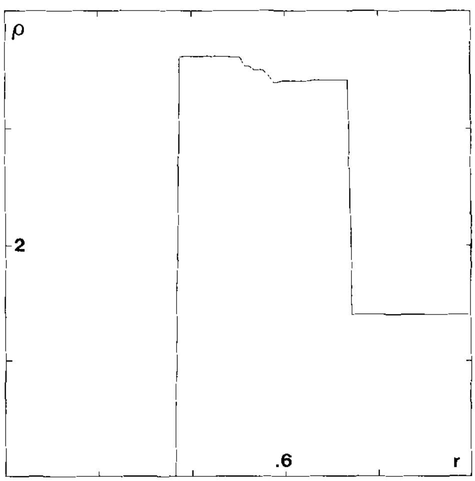

FIG. 5.7. Density versus radius at the outlet as obtained with the random choice method.

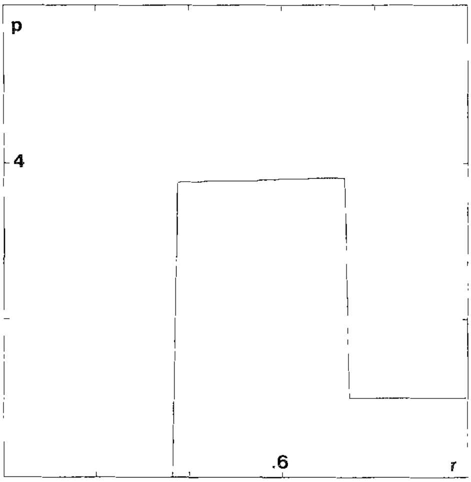

FIG. 5.8. Pressure versus radius at the outlet as obtained with the random choice method.

above and below. Now, however, the bottom boundary follows a trajectory that forms a compressive corner that turns the flow through 20° along a circular arc. Because the incoming flow is supersonic, the flow forms a compressive Prandtl–Meyer fan that develops into a shock at some distance from the boundary. For a curved boundary of very special shape, all of the Mach waves in the compressive fan meet at the same point. At this point the flow forms a Riemann problem that may be solved for the flow downstream. The solution of this Riemann problem yields a strong primary shock, a slip line, and a weak secondary wave (cf. Emanuel [15]). In our test

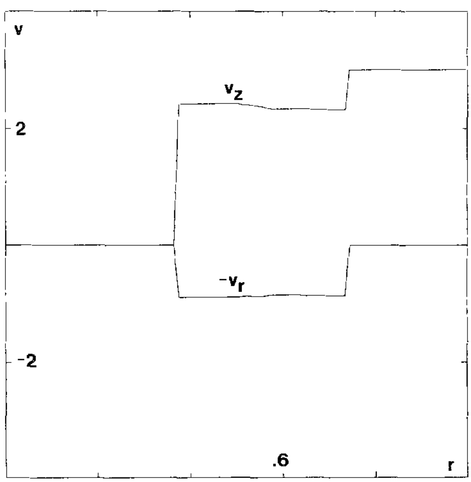

FIG. 5.9. The r and z velocities versus radius at the outlet as obtained with the random choice method.

the curved boundary is not of the special shape required, so that the intersections of the Mach waves is spread out. We would then expect that the point of shock formation is spread out, as are the slip line and secondary wave. Figures 5.6–5.9 show the results obtained using the random choice method with 100 spatial grid points. Figure 5.6 shows the isopycnics of the flow. The envelope of intersections of the Mach waves, the shock formation, and the spread-out slip line parallel to the boundary are evident. Note that because the shock is not passively tracked in this test, the density jump at the shock is spread over a number of mesh spacings by the interpolation performed by the contour plotter. The shock is in fact perfectly sharp, as seen in Figs. 5.7–5.9, which show the density, pressure, and the r and z velocities at the outlet plotted versus r, respectively. These graphs also show that the pressure and velocity are approximately constant through the spread-out slip line, so that this wave is in fact a superposition of contact discontinuities. The results obtained with this particular problem are rather remarkable in the way the weak density wave is cleanly resolved.

6. Conclusions

We have presented a random choice method for two-dimensional steady plane and axisymmetric supersonic flows. The method has been applied to various shock wave diffraction problems where it is shown that steady state solutions are correctly described. The accuracy of the method is demonstrated by way of its comparison with solutions obtained with the method of characteristics. The main advantages of the method presented are its general applicability and its sharp resolution of discontinuities.

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